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ON $\alpha\beta$-SETS

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ABSTRACT
A closed set $E \subset \mathbf{T}$ is an $\alpha\beta$-set, where α and β are elements of infinite order in $\mathbf{T}$, if $E \subset (E-\alpha) \cup (E-\beta)$. We give two constructions of "thin" $\alpha\beta$-sets.

Let α and β be elements of infinite order of the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. A closed $E \subset \mathbf{T}$ is an $\alpha\beta$-set if $E \subset (E-\alpha) \cup (E-\beta)$, or equivalently if, whenever $x \in E$, then either $x+\alpha \in E$ or $x+\beta \in E$ (or both). An $\alpha\beta$-orbit is a sequence $\{x_n\}$ such that $x_{n+1}-x_n$ is either α or β. Engelking [1] raises the question of existence of non-dense $\alpha\beta$-sets. The purpose of this note is to give a positive answer to that question.

§1

THEOREM 1. *Assume that α and β are rationally independent* (mod 1) *and that $\{y_n\}_{n=1}^{\infty} \subset \mathbf{T}$ is arbitrary. There exists a closed $\alpha\beta$-set E disjoint from $\{y_n\}$.*

REMARKS. (1) If $\{Y_n\}$ is dense in $\mathbf{T}$, the set E given by the theorem is clearly non-dense.

(2) The assumption that α and β are rationally independent is essential. If α and β are dependent, there exists some $\gamma \in \mathbf{T}$ and integers n_1, n_2 such that $\alpha = n_1\gamma$, $\beta = n_2\gamma$, and any $\alpha\beta$-orbit is a subset of bounded gaps of a γ-orbit. Since any γ-orbit is dense in $\mathbf{T}$ we see that if E is any non-empty $\alpha\beta$-set, $\bigcup_{j=0}^{M} E + j\gamma = \mathbf{T}$ where $M = \max_{j=1,2}|n_j|$ and E has non-empty interior.

For $\varepsilon > 0$ write $N(\varepsilon) = \inf(n+m)$ such that $n \geqq 0$, $m \geqq 0$, $n+m>0$ and $|n\alpha + m\beta (\mathrm{mod}\, 1)| < \varepsilon$. $N(\varepsilon)$ is clearly bounded by $1/\varepsilon$ and depends on the Diophantine properties of the pair α, β. The assumptions we made on α, β imply $\lim_{\varepsilon\to 0} N(\varepsilon) = \infty$.

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LEMMA 1. *Let E be an $\alpha\beta$-set, $E \subset \bigcup_{j=1}^{k} I_j$ where I_j are intervals of lengths $|I_j|$. Then $\Sigma_{j=1}^{k}(N(|I_j|))^{-1} \geqq 1$.*

PROOF. An $\alpha\beta$-orbit can spend no more than $(N(|I_j|))^{-1}$ of its time in I_j.

COROLLARY. *If $N(\varepsilon) > c\varepsilon^{-\eta}$ $(0 < \eta < 1)$, then the Hausdorff dimension of every $\alpha\beta$-set is at least η.*

We shall use the following variant of Lemma 1:

LEMMA 2. *Assume that $\{I_j\}_{j=1}^{k}$ are open intervals on $\mathbf{T}$ such that $\Sigma_{j=1}^{k} N(|I_j|)^{-1} < 1/10$. There exists an integer K such that if $\{x_l\}_{l=0}^{M}$ is an $\alpha\beta$-orbit and $x_l \in \bigcup I_j$ for $\frac{1}{2}M$ values of l, then $M < K$.*

PROOF. Long orbits can spend no more the 1/10 of their time in $\bigcup I_j$.

PROOF OF THEOREM 1. We construct E by removing from $\mathbf{T}$, successively, finite unions of open intervals and taking the intersection of whatever is left. The first stage consists of removing from $\mathbf{T}$ an open interval I_1, containing y_1, such that $N(|I_1|)^{-1} < 1/10$, where $|I_1|$ is also small enough to insure that $E_1 = \mathbf{T} \setminus I_1$ is an $\alpha\beta$-set. We also require that no end point of I_1 (as well as those of any interval which we remove later) belongs to the same $\alpha\beta$-orbit as any of the points $\{y_j\}$.

After n steps we obtain a set $E_n = \mathbf{T} \setminus \bigcup_{j=1}^{k_n} I_j$ which is an $\alpha\beta$-set and such that

$$\sum_{j=1}^{k_n} N(|I_j|)^{-1} < 1/10 \tag{1}$$

and $\{y_j\}_{j=1}^{n} \subset I_j$. Let y' be the first y_j not covered by $\bigcup I_j$. It is clear that if we remove y' from E_n and we are to obtain an $\alpha\beta$-set, we have to remove at the same time all the points $x \in E_n$ such that any $\alpha\beta$-orbit starting at x and remaining in E_n must pass through y'. Such points x clearly have the form $y' = k\alpha - l\beta$, $k \geqq 0$, $l \geqq 0$, $k + l > 0$, and we claim that there are only a finite number of those, in fact, that $k + l < K$ where K is the integer given by Lemma 2 for $\{I_j\}_{j=1}^{k_n}$. Let $x = y' - k\alpha - l\beta$ be such a point and assume that $l \geqq k$; any $\alpha\beta$-orbit from x which remains in E_n must have k α-steps and l β-steps in its first $k + l$ steps. Here we use the independence of α and β and the fact that such an orbit must pass through y'. Of the usually several such orbits from x to y' there is one that favors α in the sense that whenever there is an option to step α or β, α is done. Denote this orbit by $\{\xi_j\}_{j=0}^{k+l}$ $(\xi_0 = x,\ \xi_{k+l} = y')$; there are l values of j for which the β-step was forced, i.e., $\xi_j + \alpha \notin E_n$. If we look at the orbit $\{\xi_j + \alpha\}_{j=0}^{k+l}$ we see that l of its $k + l$ members are in $\bigcup_{1}^{k_n} I_j$ and since $l \geqq \frac{1}{2}(k + l)$

we can apply Lemma 2 and obtain $k+l \leqq K$. If $l<k$ interchange the roles of α and β and again $k+l \leqq K$. Notice that the bound K depends only on the structure of E_n and not on the point y'. If we denote by $R(y')$ the set of points which have to be removed with y', we see that this set is contained in the interior of E_n (by the assumption made on the boundary points) and clearly depends continuously on y' in the sense that for all sufficiently small δ we have $r(y'+\delta)=R(y')+\delta$. Thus, if we remove a δ neighborhood of $R(y')$, that is the union of intervals of length 2δ centered at each $x \in R(y')$, the remaining set E_{n+1} will again be an $\alpha\beta$-set. We impose on δ also to be small enough so that (1) be valid for $n+1$, and finally that the new boundary points not share $\alpha\beta$-orbits with any y_j.

§2. The construction described above clearly gives sets of positive measure. It seems probable that for any independent pair α, β there exist $\alpha\beta$-sets of measure zero. On the other hand, the corollary to Lemma 1 shows that in general $\alpha\beta$-sets have positive Hausdorff dimension. Our next construction shows that for some pairs α, β there exist extremely thin $\alpha\beta$-sets.

THEOREM 2. *Given any Hausdorff function h, there exist pairs α, β and $\alpha\beta$-sets E such that Hausdorff h-measure of E is zero.*

PROOF. We construct α, β and E simultaneously by successive approximation of α and β by rationals and E by finite sets. The thinner we want E the more "Liouville" will α and β have to be. We start by putting $\alpha_1 = 0.1$, $\beta_1 = 0.3$ and $E = \{0, \alpha_1, 2\alpha_1, 3\alpha_1, 4\alpha_1 = \alpha_1+\beta_1, 7\alpha_1 = \alpha_1 + 2\beta_1\}$. We have two closed orbits in E_1 starting at zero, namely,

a_1: four α_1-steps followed by two β_1-steps,

b_1: one α_1-step followed by three β_1-steps.

We use the notation a_1, b_1 (and similarly a_n, b_n etc. later) as lists of steps to take. Thus, for any α^*, β^*, $a_1(\alpha^*, \beta^*)$ will mean: "take four α^*-steps and then two β^*-steps".

Let N_2 and M_2 be (large) positive integers. For every ε_2, η_2 we can solve the equations $4x+2y=1+\varepsilon_2$, $x+3y=1+\eta_2$. If ε_2 and η_2 are both small, the solution (α_2, β_2) will be close to (α_1, β_1). We take $\varepsilon_2 > 0$, $\eta_2 > 0$ such that $\eta_2 = N_2\varepsilon_2$, η_2 is roughly $(M_2+1)^{-1}\alpha_1$ and adjusted so that the following picture holds: Starting from zero do $a_1(\alpha_2, \beta_2)$; we overshot zero by ε_2. Repeat $a_1(\alpha_2, \beta_2)$ N_2 additional times. We now have N_2+1 copies of (the range of) a_1, each a translate by ε_2 of the preceding. Continue by doing $b_1(\alpha_2, \beta_2)$ M_2 times. We are now just short of α_2. Continue with $b_1^*(\alpha_2, \beta_2)$, where b_1^* is b_1 *from which we omit* *the first step*, and we are back at zero. We have just described the orbit $a_2(\alpha_2, \beta_2)$. The orbit $b_2(\alpha_2, \beta_2)$ is: a single a_1 orbit followed by $M_2 + 1$ b_1 orbits and closing with one b_1^*. In other words we replace in a_2, N_2 orbits a_1 by a single orbit b_1; this is possible since $N_2\varepsilon_2 = \eta_2$. For use in the following step we also define b_2^* as b_2 omitting the first a_1, that is, $M_2 + 1$ orbits b_1 followed by one b_1^*. Define E_2 as the set covered by $a_2(\alpha_2, \beta_2)$ and $b_2(\alpha_2, \beta_2)$, both starting (and ending) at zero. We have $E_2 \subset E_1 + [-\eta_2, \alpha_1]$. If we denote by k_2 (resp. k_2') the number of α-steps in a_2 (resp. b_2) and by l_2 (resp. l_2') the corresponding number of β-steps, we have $k_2 = 4(N_2 + 1) + M_2$, $k_2' = M_2 + 5$, $l_2 = 2(N_2 + 1) + 3(M_2 + 1)$, $l_2' = 3(M_2 + 2) + 2$; and if we impose that $N_2 \gg M_2$ we obtain

$$\begin{vmatrix} k_2 & l_2 \\ k_2' & l_2' \end{vmatrix} \sim N_2 M_2 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} \neq 0 \tag{2}$$

and we are set to continue by induction.

The typical step in the induction is completely analogous to the one just described. We choose $N_j \gg M_j$ and thereby ε_j and $\eta_i = N_j\varepsilon_j \sim (M_j + 1)^{-1}\varepsilon_{j-1}$. We obtain (α_j, β_j) very close to $(\alpha_{j-1}, \beta_{j-1})$ as the solution of

$$\begin{aligned} k_j(\alpha_j - \alpha_{j-1}) + l_j(\beta_j - \beta_{j-1}) &= \varepsilon_j, \\ k_j'(\alpha_j - \alpha_{j-1}) + l_j'(\beta_j - \beta_{j-1}) &= \eta_j \end{aligned} \tag{3}$$

and define $a_j = (a_{j-1} : N_j + 1$ times, $b_{j-1} : M_j$ times and one $b_{j-1}^*)$, $b_j = (a_{j-1}$ followed by $M_j + 1$ times b_{j-1} and one $b_{j-1}^*)$. We select η_j so that $a_j(\alpha_j, \beta_j)$ and $b_j(\alpha_j, \beta_j)$ both close at their initial point. We put $b_j^* = (b_{j-1} : M_j + 1$ times and one $b_{j-1}^*)$ and E_j the set of points covered by either $a_j(\alpha_j, \beta_j)$ or $b_j(\alpha_j, \beta_j)$ both starting at zero. As in (2), we insure the solvability of (3) by choosing N_{j-1} and M_{j-1} large enough. We now write $(E, \alpha, \beta) = \lim_{j\to\infty}(E_j, \alpha_j, \beta_j)$. In order to estimate the Hausdorff h-measure we notice again that $E \subset E_j + [-2\eta_{j+1}, 2\varepsilon_j]$. We denote the total number of points in E_j by P_j and remark that by taking M_{j+1} large enough, η_{j+1} becomes small enough to insure $P_j h(4\eta_{j+1}) < j^{-1}$. We cover part of E by $F_j = E_j + [-2\eta_{j+1}, 2\eta_{j+1}]$ and notice that the number of points of E_{j+1} not covered by F_j is bounded by $P_j M_{j+1}$ which is independent of N_{j+1}. We now take N_{j+1} large enough, i.e. ε_{j+1} small enough to insure $P_j M_{j+1} h(4\varepsilon_{j+1}) < j^{-1}$ and the theorem follows from $E \subset F_j \cup [(E_{j+1} \backslash F_j) + [-2\varepsilon_{j+1}, 2\varepsilon_{j+1}]]$.

Reference

1. R. Engelking, *Sur un problème de K. Urbanik*, Colloq. Math. **8** (1961), 243–250.

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